Recursive Utility–Epstein-Zin Preferences

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Our goal is to derive the stochastic discount factor (SDF) when the representative agent has Epstein-Zin (EZ) preferences.[1]

1 Preliminaries

In general, an individual has *recursive preferences* if utility satisfies the recursion

$$V_t = F(C_t, R(V_{t+1})),$$

where $R(\cdot)$ is a function performing a risk adjustment that helps to compute the certain equivalence of future utility, and $F(\cdot)$ is a time aggregator capturing preferences over the timing of consumption. Note that a particular case would be the one of time separable utility, which can be written as $V_t = u(C_t) + \beta \mathbb{E}_t(V_{t+1})$, for some function $u(\cdot)$.

The EZ preferences are a particular case of recursive preferences in which there is a CES time aggregator and CRRA risk adjustment. Letting $u(C_t)$ to be the utility flow of consumption and V_t the lifetime expected utility, then we have

$$V_t = \mathbb{E}_t \sum_{j=0}^{\infty} u(C_{t+j}) = u(C_t) + \beta \mathbb{E}_t(V_{t+1}). \tag{1}$$

Writing Eq. (1) in CES form

$$V_t = \left[(1-\beta) u(C_t)^{1-\rho} + \beta \Big(\mathbb{E}_t (V_{t+1})^{1-\rho} \Big) \right]^{\frac{1}{1-\rho}},$$

where ρ is the inverse of the intertemporal elasticity of substitution, and applying the CRRA risk adjustment

$$V_t = \left[(1-\beta) u(C_t)^{1-\rho} + \beta \Big(\mathbb{E}_t (V_{t+1}^{1-\gamma})^{\frac{1-\rho}{1-\gamma}} \Big) \right]^{\frac{1}{1-\rho}}, \tag{2}$$

we get the EZ recursion. There are three observations about Eq. (2)

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[1] These notes are based on the lecture notes of Nelson Mark and Dejanir Silva.

1. The standard CRRA case follows when $\gamma = \rho$.
2. The recursion is well defined if $u(\cdot) \geq 0$ everywhere and $V_t \geq 0$. If that is not the case so $u(\cdot) \leq 0$ holds, we let $V_t \leq 0$ and define the recursion as

$$V_t = \left[(1-\beta)u(C_t)^{1-\rho} - \beta\Big(\mathbb{E}_t(-\{V_{t+1}\}^{1-\gamma})^{\frac{1-\rho}{1-\gamma}}\Big)\right]^{\frac{1}{1-\rho}}.$$

3. The case $\gamma > \rho$ corresponds to a preference for early resolution of uncertainty.

2 Deriving the SDF

Note that Eq. (2) can be written as

$$V_t = F(C_t, R(V_{t+1})) = \left[(1-\beta)u(C_t)^{1-\rho} + \beta\Big(R(V_{t+1})\Big)^{1-\rho}\right]^{\frac{1}{1-\rho}}, \tag{3}$$

with $R(V_{t+1}) = \Big(\mathbb{E}_t(V_{t+1}^{1-\gamma})\Big)^{\frac{1}{1-\gamma}}$.
The partial derivative with respect to current consumption is

$$\frac{\partial V_t}{\partial C_t} = (1-\beta)\left(\frac{V_t}{C_t}\right)^{\rho}.$$

On the other hand, the partial derivative with respect to future consumption is a highly nonlinear object. Therefore, we proceed with the following chain of partial derivatives

$$\frac{\partial V_t}{\partial C_{t+1}} = \frac{\partial V_t}{\partial R(V_{t+1})} \times \frac{\partial R(V_{t+1})}{\partial V_{t+1}^{1-\gamma}} \times \frac{\partial V_{t+1}^{1-\gamma}}{\partial V_{t+1}} \times \frac{\partial V_{t+1}}{\partial C_{t+1}}$$

with

$$\begin{aligned}
\frac{\partial V_t}{\partial R(V_{t+1})} &= \beta\left(\frac{V_t}{R(V_{t+1})}\right)^{\rho} \\
\frac{\partial R(V_{t+1})}{\partial V_{t+1}^{1-\gamma}} &= \frac{\partial\Big(\mathbb{E}_t(V_{t+1}^{1-\gamma})\Big)^{\frac{1}{1-\gamma}}}{\partial \mathbb{E}_t(V_{t+1}^{1-\gamma})} \times \frac{\partial \mathbb{E}_t(V_{t+1}^{1-\gamma})}{\partial V_{t+1}^{1-\gamma}} \times \frac{\partial V_{t+1}^{1-\gamma}}{\partial V_{t+1}} \\
&= \frac{1}{1-\gamma} R(V_{t+1})^{\gamma} \times 1 \times (1-\gamma)V_{t+1}^{-\gamma} = \left(\frac{R(V_{t+1})}{V_{t+1}}\right)^{\gamma} \\
\frac{\partial V_{t+1}^{1-\gamma}}{\partial V_{t+1}} &= (1-\gamma)V_{t+1}^{-\gamma} \\
\frac{\partial V_{t+1}}{\partial C_{t+1}} &= (1-\beta)\left(\frac{V_{t+1}}{C_{t+1}}\right)^{\rho},
\end{aligned}$$

where the last expression uses the partial derivative with respect to current consumption, iterated one period forward.

Combining all the previous elements we have

$$\frac{\partial V_t}{\partial C_{t+1}} = \beta(1-\beta)V_t^{\rho} C_{t+1}^{-\rho} \left(\frac{V_{t+1}}{R(V_{t+1})} \right)^{\rho-\gamma}.$$

The SDF between period t and $t+1$ reads as

$$\begin{aligned} M_{t,t+1} &= \frac{\partial V_t / \partial C_{t+1}}{\partial V_t / \partial C_t} \\ M_{t,t+1} &= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{\left(\mathbb{E}_t (V_{t+1}^{1-\gamma}) \right)^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma}. \end{aligned}$$